Lagrangian formulation of irreducible massive fields of arbitrary spin in 2+1 dimensions

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Lagrangian formulation of free massive fields corresponding to irreducible representations of the Poincare group of arbitrary integer and half–integer spins in three–dimensional space–time is presented. A relationship of the theory under consideration with the self–duality equations in four dimensions is discussed.

1 Introduction

A purpose of the present paper is to give a Lagrangian formulation of free massive fields corresponding to an irreducible representation of the Poincare group of an arbitrary integer (bosons) or half–integer (fermions) spin in the three-dimensional space-time. An interest to lower dimensional models is due to the fact that these relatively simple theories serve as a polygon for study of more realistic models in higher-dimensional spaces. Also one can hope that such models can describe real film–type physical systems.

After the classical work by Fierz and Pauli [1], there was a great number of papers in which gauge invariant actions were formulated for massless fields of an arbitrary spin in any dimension [2] as well as actions for massive fields corresponding to totally symmetric representations of the little group in dimensions $D \geq 4$ [3]. In the work [4], it was demonstrated that the actions for massive fields in D dimensions can be derived by means of the dimensional reduction of the actions for massless fields in $D + 1$ dimensions. This automatically gives rise to supplementary fields necessary for the formulation of a local action. A straightforward application of such a procedure to a derivation of the actions of massive fields in 3 dimensions via reduction of the gauge invariant action in 4 dimensions [5, 6, 7, 8, 9] faces

however the following problem. Gauge invariant actions in 4 dimensions describe fields possessing two helicities (these fields can be interpreted either as Majorana fields with two values of helicity or as Weyl fields and antifields, each having a fixed value of helicity). Dimensional reduction to the three–dimensional space gives rise to an action which describes a field with two values of the (2+1)–dimensional spin, which therefore constitutes some reducible representation of the (2+1)–dimensional Poincare group [10]. Since it is impossible to build a local action in four dimensions which describes a self–dual field possessing a unique value of helicity [11] (i.e. Majorana-Weyl field), after dimensional reduction $4 \rightarrow 3$ it is still necessary to carry out an additional reduction which leaves a half of physical degrees of freedom if one wishes to have a physical system which corresponds to an irreducible representation of the Poincare group in 2+1 dimensions.

So far the actions for irreducible massive fields in (2+1)–dimensional space with spins $s = 1$ [12, 13, 14], $s = 3/2$ [15, 16], $s = 2$ [14, 17] and s=5/2 [18, 19] have been constructed. The formalism which we apply in this paper for the Lagrangian description of arbitrary irreducible representations of (2+1)– dimensional Poincare group is related to the reduction to 2+1 dimensions of the vierbein-type variables introduced for the description of massless fields of an arbitrary integer [8] and half-integer [8, 9] spin in 3+1 dimensions. For the cases of spins 3/2, 2 and 5/2 we reproduce the results obtained in [16], [14] and [18], respectively.

A structure of the paper is as follows. In the section 2 we summarize the main notations used throughout the paper. In the section 3 the actions, which are of first order in derivatives, are constructed for boson fields of an arbitrary integer spin s, $|s| \geq 2$. In the section 4 an analogous problem is solved for fermion fields of an arbitrary half-integer spin s, $|s| \geq 3/2$. The conditions on the parameters of the model (i.e. sign choices in front of the kinetic and mass terms in the action) which follow from the requirements of boundness of energy and positive definiteness of norms of states are derived in the section 5. In section 6, an action is presented for the bosonic case of integer spin s, which is of the second order in derivatives and describes the both irreducible representations of the (2+1)–dimensional Poincare group for a fixed value of $|s|$.

2 Notations and Problem Setting

Our field variables include the fields $\lambda^n_{\alpha(n)} \equiv \lambda^n_{\alpha_1 \ldots \alpha_n}$ and $h^n_{\mu|\alpha(n)} \equiv h^n_{\mu|\alpha_1 \ldots \alpha_n}$, which are totally symmetric in spinor indices $(\mu, \nu, \ldots = 0, 1, 2, \quad \alpha, \beta, \ldots = 1, 2)^*$. Also we will use the shorthand notations λ^n and h^n_μ for these fields. Vectorial indices are raised and lowered as usual by virtue of the metric tensors $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(+, -, -)$. Spinorial indices are raised and lowered by virtue of $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ and $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon_{12} = \varepsilon^{12} = 1$:

$$A_\alpha = A^\beta \varepsilon_{\beta\alpha}, \quad A^\alpha = \varepsilon^{\alpha\beta} A_\beta, \quad A^\alpha B_\alpha = -A_\alpha B^\alpha.$$

The Lorentz transformations of fields h^n_μ and λ^n, which correspond to the transformation

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*Greek indices from the first part of the Alphabet are spinorial while Greek indices from the second part of the alphabet are vectorial.

of the coordinates $\delta x^\mu = \omega^{\mu\nu} x_\nu$, $\omega^{\mu\nu} = -\omega^{\nu\mu}$, have a form:

$$\delta h^n_{\mu|\alpha(n)} = \frac{i}{2}\omega^{\nu\sigma}(M_{\nu\sigma}h^n)_{\mu|\alpha(n)} =$$

$$\omega^{\nu\sigma}x_\nu\partial_\sigma h^n_{\mu|\alpha(n)} + \omega_{\mu\nu}h^{n\nu|}{}_{\alpha(n)} - \frac{n}{4}\omega^{\nu\sigma}\varepsilon_{\nu\sigma\rho}e^{\rho|}{}_\alpha{}^\beta h^n_{\mu|\alpha(n-1)\beta}, \tag{1}$$

$$\delta\lambda^n_{\alpha(n)} = \omega^{\nu\sigma}x_\nu\partial_\sigma\lambda^n_{\alpha(n)} - \frac{n}{4}\omega^{\nu\sigma}\varepsilon_{\nu\sigma\rho}e^{\rho|}{}_\alpha{}^\beta\lambda^n_{\alpha(n-1)\beta}. \tag{2}$$

Here $\partial_\mu \equiv \partial/\partial x^\mu$, $\varepsilon_{\mu\nu\sigma}$ is the totally antisymmetric tensor $\varepsilon_{012} = \varepsilon^{012} = 1$, while $e^{\mu|}{}_{\alpha\beta}$ is a set of three real symmetric matrices which are defined by the relation

$$e^{\mu|}{}_{\alpha\gamma}e^{\nu|}{}_\beta{}^\gamma = \eta^{\mu\nu}\varepsilon_{\alpha\beta} - \varepsilon^{\mu\nu\sigma}e_{\sigma|\alpha\beta}$$

and satisfy the following useful identities:

$$e_{\mu|\alpha(2)}e^{\mu|\beta(2)} = 2\delta^\beta_\alpha\delta^\beta_\alpha, \quad \varepsilon^{\mu\nu\sigma}e_{\nu|\alpha(2)}e_\sigma{}^{|\beta(2)} = -2\delta^\beta_\alpha e^{\mu|}{}_\alpha{}^\beta$$

(in the section 6 we will use a special realization with $e^{0|}{}_{\alpha\beta} = \delta_{\alpha\beta}$). We use the following convention [20] :

$$A^{k+l}_{\alpha(k)\beta(l)}B^{m+n}_{\alpha(m)\gamma(n)} \equiv \frac{1}{C^k_{k+m}}\sum A^{k+l}_{\alpha_{i_1}...\alpha_{i_k}\beta(l)}B^{m+n}_{\alpha_{i_{k+1}}...\alpha_{i_{k+m}}\gamma(n)},$$

where the summation is carried out over all C^k_{k+m} combinations of indices α_i (A^{k+l} and B^{m+n} are symmetric with respect to their spinorial indices.) Also

$$A^{n+m\alpha(n)\beta(m)}B^{n+k+l}_{\alpha(n+k)\gamma(l)} \equiv \sum_{\alpha_i} A^{n+m\alpha_1...\alpha_n\beta(m)}B^{n+k+l}_{\alpha_1...\alpha_n\alpha(k)\gamma(l)},$$

$$A^{n+m\alpha(n)\beta(m)}B^{n_1+k}_{\alpha(n_1)\gamma(k)}C^{n-n_1+l}_{\alpha(n-n_1)\delta(l)} = A^{n+m\alpha(n)\beta(m)}\left(B^{n_1+k}_{\alpha(n_1)\gamma(k)}C^{n-n_1+l}_{\alpha(n-n_1)\delta(l)}\right).$$

The Casimir operator of the Poincare group (the Pauli–Lubanski scalar) W,

$$W = -\frac{i}{2}\varepsilon^{\mu\nu\sigma}M_{\nu\sigma}\partial_\mu,$$

acts on h^n_μ according to the rule:

$$(Wh^n)_{\mu|\alpha(n)} = \varepsilon_\mu{}^{\nu\sigma}\partial_\nu h^n_{\sigma|\alpha(n)} + \frac{n}{2}e^{\sigma|}{}_\alpha{}^\beta\partial_\sigma h^n_{\mu|\alpha(n-1)\beta}. \tag{3}$$

Its eigenvalues ms determine a spin s of a state with respect of the Poincare group while m is a mass of a state which will be always assumed to be positive.

A field h^N_μ describes a single (highest) spin $\pm(N+2)/2$ if it satisfies the conditions,

$$e^{\mu|}{}_\beta{}^\gamma h^N_{\mu|\alpha(N-1)\gamma} = 0 \tag{4}$$

(which implies that the field $\tilde{h}^{N+2}_{\beta_1\beta_2\alpha_1...\alpha_N} \equiv (1/2)e^{\mu|}{}_{\beta_1\beta_2}h^N_{\mu|\alpha_1...\alpha_N}$ is symmetric with respect to all spinor indices) and, e.g.

$$\varepsilon_\mu{}^{\nu\sigma}\partial_\nu h^N_{\sigma|\alpha(N)} = \pm mh^N_{\mu|\alpha(N)} \tag{5}$$

which implies that the field $\tilde{h}^{N+2}$ satisfies a Dirac equation with respect to each spinorial index.

Our goal is to derive an extremal principle which gives rise to the equations (4), (5) and, may be, to some dynamically trivial equations for supplementary fields.

3   Boson Fields

In this section we construct an action for boson fields of an arbitrary integer spin s, $|s| = (N + 2)/2 \geq 2$, $N \geq 2$ – even. Let us start with the action:

$$S_B = \frac{1}{2}\sum_{n=0}^{N} \xi_n \varepsilon^{\mu\nu\sigma} h_{\mu|}^{n\ \alpha(n)} \partial_\nu h_{\sigma|\alpha(n)}^n + \frac{m}{2}\sum_{n=2}^{N} a_n \varepsilon^{\mu\nu\sigma} h_{\mu|}^{n\ \alpha(n-1)\beta} e_{\nu|\beta}{}^\gamma h_{\sigma|\alpha(n-1)\gamma}^n +$$

$$+ m\sum_{n=2}^{N} b_n \varepsilon^{\mu\nu\sigma} h_{\mu|}^{n\ \alpha(n)} e_{\nu|\alpha(2)} h_{\sigma|\alpha(n-2)}^{n-2} + m\sum_{n=0}^{N-2} \lambda_{\alpha(n+2)}^{n+2} e^{\mu|\alpha(2)} h_{\mu|}^{n\ \alpha(n)}, \tag{6}$$

where summation is carried out over even n. Boson fields h_μ^n, λ^n are real. From the requirement that the action has to be real it follows that the coefficients ξ_n, a_n, b_n should be real. We assume that $\xi_n = \pm 1$ that can always be achieved by an appropriate rescaling of fields. The action (6) is invariant under the Lorentz transformations (1), (2). The problem consists of choosing such coefficients ξ_n, a_n, b_n that the extra coefficients of the field h_μ^N (including one of the highest spins) vanish as a result of field equations, as well as all fields h_μ^n with $0 \leq n \leq N - 2$, and λ^n, $2 \leq n \leq N$.

The variation of the action with respect to λ^n gives the constraints

$$e^{\mu|}{}_{\alpha(2)} h_{\mu|\alpha(n)}^n = 0, \quad 0 \leq n \leq N - 2. \tag{7}$$

From (7) at $n = 0$ it follows that

$$h_\mu^0 = 0. \tag{8}$$

The variation of the action over h_μ^n gives the equations of motion:

$$\xi_n \varepsilon^{\mu\nu\sigma} \partial_\nu h_{\sigma|\alpha(n)}^n + (1 - \delta_{n,0}) m a_n \varepsilon^{\mu\nu\sigma} e_{\nu|\alpha}{}^\beta h_{\sigma|\alpha(n-1)\beta}^n + (1 - \delta_{n,0}) m b_n \varepsilon^{\mu\nu\sigma} e_{\nu|\alpha(2)} h_{\sigma|\alpha(n-2)}^{n-2} -$$

$$(1 - \delta_{n,N}) m b_{n+2} \varepsilon^{\mu\nu\sigma} e_{\nu|}{}^{\alpha(2)} h_{\sigma|\alpha(n+2)}^{n+2} + (1 - \delta_{n,N}) m e^{\mu|\alpha(2)} \lambda_{\alpha(n+2)}^{n+2} = 0. \tag{9}$$

Let us contract the equations (9) with ∂_μ and exclude derivatives of all functions h_σ^n in the resulting equations with the aid of (9). For every n one is then left with some terms containing h_σ^{n+4}, h_σ^{n+2}, h_σ^n, h_σ^{n-2} and h_σ^{n-4}. The coefficients in front of h_σ^{n+4} and h_σ^{n-4} vanish identically due to the properties of the matrices e_μ and the antisymmetrization over vectorial indices. A term with h_σ^{n-2} vanishes as a consequence of the constraints (7).[†] As a result one derives after some transformations:

$$2\xi_2 a_2 b_2 e^{\mu|\alpha(2)} h_{\mu|\alpha(2)}^2 = \frac{1}{m} e^{\mu|\alpha(2)} \partial_\mu \lambda_{\alpha(2)}^2, \tag{10}$$

$$b_{n+2}\left(\xi_n a_n - \frac{n+4}{n+2}\xi_{n+2} a_{n+2}\right) e^{\sigma|\alpha(2)} h_{\sigma|\alpha(n+2)}^{n+2} +$$

$$\left(\frac{2}{n}\xi_n a_n^2 + 2\xi_{n-2} b_n^2 - \frac{2n(n+3)}{(n+1)(n+2)}\xi_{n+2} b_{n+2}^2\right) e^{\sigma|}{}_\alpha{}^\beta h_{\sigma|\alpha(n-1)\beta}^n +$$

[†]Note that at this point only one of the constraints is really necessary, namely (8). For $n \geq 4$ the coefficients in front of h_σ^{n-2} vanish provided that a_n have a form (12).

$$2\xi_{n-2}b_n\lambda^n_{\alpha(n)} + \frac{1}{m}(1 - \delta_{n,N})e^{\mu|\alpha(2)}\partial_\mu\lambda^{n+2}_{\alpha(n+2)} = 0, \quad 2 \leq n \leq N, \tag{11}$$

where, by definition, $b_{N+2} = 0$.

Let us require the coefficients in front of h^{n+2}_σ and h^n_σ in (11) to vanish:

$$\xi_n a_n - \frac{n+4}{n+2}\xi_{n+2}a_{n+2} = 0, \quad 2 \leq n \leq N,$$

$$\frac{1}{n}a^2_n + \xi_{n-2}b^2_n - \frac{n(n+3)}{(n+1)(n+2)}\xi_{n+2}b^2_{n+2} = 0, \quad 2 \leq n \leq N.$$

A solution of these equations reads:

$$a_n = \frac{N+2}{n+2}\xi_N\xi_n(-1)^{(N+2)/2}\theta, \quad b^2_n = -\xi_{n-2}\xi_n\frac{(N+2)^2 - n^2}{4n(n+1)}, \quad \theta^2 = 1,$$

where we have chosen a normalization $a_N = (-1)^{(N+2)/2}\theta$, which can always be achieved by virtue of some redefinition of the parameter m. In the section 5 it will be shown that the requirements of boundness of energy and positive definiteness of the norms of states is satisfied only for $\theta = 1$ which condition is assumed to be true from now on. The condition $b^2_n > 0$ gives $\xi_{n-2}\xi_n = -1$, i.e.

$$\xi_n = (-1)^{(n+2)/2}\xi, \quad \xi^2 = 1,$$

so that one finally gets

$$\xi_n = (-1)^{(n+2)/2}\xi, \quad a_n = (-1)^{(n+2)/2}\frac{N+2}{n+2}, \quad b_n = \frac{(-1)^{(n+2)/2}}{2}\sqrt{\frac{(N+2)^2 - n^2}{n(n+1)}}. \tag{12}$$

Any other choice of signs for b_n can be compensated by a simple field redefinition.

Setting $n = N$ in (11), we obtain

$$\lambda^N_{\alpha(N)} = 0. \tag{13}$$

Setting then $n = N - 2$ in (11) and taking into account (13), one gets

$$\lambda^{N-2}_{\alpha(N-2)} = 0,$$

etc.

Finally, from (11) it follows that all Lagrange multipliers vanish as a consequence of the equations of motion

$$\lambda^n_{\alpha(n)} = 0, \quad n = 2, \ldots, N.$$

Let us now analyze the equations for h^n_μ.

Let $N = 2$.

The equation of motion (9) at $n = 0$ (taking into account (8)) along with the equation (10) give

$$\varepsilon^{\mu\nu\sigma}e_{\nu|}{}^{\alpha(2)}h^2_{\sigma|\alpha(2)} = 0, \quad e^{\mu|\alpha(2)}h^2_{\mu|\alpha(2)} = 0,$$

that is equivalent to the condition (4). The equation of motion (9) at $n = 2$ has a form

$$\varepsilon^{\mu\nu\sigma}\partial_\nu h^2_{\sigma|\alpha(2)} + \xi m \varepsilon^{\mu\nu\sigma} e_{\nu|\alpha}{}^\beta h^2_{\sigma|\alpha\beta} = 0,$$

which is equivalent to (5).

Let $N > 2$.

The equation of motion (9) at $n = 0$ (taking into account (8)), the constraint (7) at $n = 2$ and the equation (10) give

$$\varepsilon^{\mu\nu\sigma} e_{\nu|}{}^{\alpha(2)} h^2_{\sigma|\alpha(2)} = 0, \quad e^{\mu|}{}_{\alpha(2)} h^2_{\mu|\alpha(2)} = 0, \quad e^{\mu|\alpha(2)} h^2_{\mu|\alpha(2)} = 0.$$

It is not difficult to see that from these relations it follows that

$$h^2_{\mu|\alpha(2)} = 0.$$

Next, let us assume that it is shown that all h^k_μ, $2 \le k \le n_0 < N - 2$, vanish as a consequence of the equations of motion. Then the equation of motion (9) at $n = n_0$ along with the constraint (7) at $n = n_0 + 2$ give

$$\varepsilon^{\mu\nu\sigma} e_{\nu|}{}^{\alpha(2)} h^{n_0+2}_{\sigma|\alpha(n_0+2)} = 0, \quad e^{\mu|}{}_{\alpha(2)} h^{n_0+2}_{\mu|\alpha(n_0+2)} = 0,$$

from where it follows that

$$h^{n_0+2}_{\mu|\alpha(n_0+2)} = 0.$$

Thus the equation of motion (9) at $n \le N - 4$, the constraints (8) and (7) along with the equations (10) give:

$$h^n_{\mu|\alpha(n)} = 0, \quad n = 0, \ldots, N - 2.$$

Finally, the equations of motion (9) at $n = N - 2$ and $n = N$ give:

$$\varepsilon^{\mu\nu\sigma} e_{\nu|}{}^{\alpha(2)} h^N_{\sigma|\alpha(N)} = 0, \tag{14}$$

$$\varepsilon^{\mu\nu\sigma}\partial_\nu h^N_{\sigma|\alpha(N)} + \xi m \varepsilon^{\mu\nu\sigma} e_{\nu|\alpha}{}^\beta h^N_{\sigma|\alpha(N-1)\beta} = 0. \tag{15}$$

The condition (14) at $N \ge 3$ is equivalent to the condition (4). As a consequence of this condition the equations (15) can be re-written in the following equivalent forms:

$$\varepsilon_\mu{}^{\nu\sigma}\partial_\nu h^N_{\sigma|\alpha(N)} - \xi m h^N_{\mu|\alpha(N)} = 0, \tag{16}$$

$$e^{\nu|}{}_\beta{}^\gamma \partial_\nu h^N_{\mu|\alpha(N-1)\gamma} - \xi m h^N_{\mu|\beta\alpha(N-1)} = 0.$$

From these equations it follows:

$$(\Box + m^2) h^N_{\mu|\alpha(N)} = 0, \quad \Box = \partial_0^2 - \partial_1^2 - \partial_2^2,$$

$$(W h^N)_{\mu|\alpha(N)} = m \left(\xi \frac{N+2}{2} \right) h^N_{\mu|\alpha(N)}. \tag{17}$$

As a result, the parameter m coincides with the rest mass while from (17) it follows that the field h^N_μ has a spin $\xi(N+2)/2$.

Thus it is shown that the action (6) with the parameters (12) describes a spin $\xi(N+2)/2$ irreducible representation of the Poincare group in 2+1 dimensions.

4 Fermionic Fields

In this section we derive an action for fermionic fields of an arbitrary half-integer spin s, $|s| = (N + 2)/2 \geq 3/2$, $N \geq 1$ is odd:

$$S_F = \frac{i}{2} \sum_{n=1}^{N} \xi_n \varepsilon^{\mu\nu\sigma} h_{\mu|}^{n\ \alpha(n)} \partial_\nu h_{\sigma|\alpha(n)}^n + \frac{i\varepsilon m}{2} \sum_{n=1}^{N} a_n \varepsilon^{\mu\nu\sigma} h_{\mu|}^{n\ \alpha(n-1)\beta} e_{\nu|\beta}{}^\gamma h_{\sigma|\alpha(n-1)\gamma}^n +$$

$$im \sum_{n=3}^{N} b_n \varepsilon^{\mu\nu\sigma} h_{\mu|}^{n\ \alpha(n)} e_{\nu|\alpha(2)} h_{\sigma|\alpha(n-2)}^{n-2} + im \sum_{n=1}^{N-2} \lambda_{\alpha(n+2)}^{n+2} e^{\mu|\alpha(2)} h_{\mu|}^{n\ \alpha(n)}, \tag{18}$$

where summation is carried out over odd n. Fermionic fields h_μ^n and λ^n are real as well as the coefficients a_n and b_n, $\xi_n = \pm 1$, $\varepsilon = \pm 1$.

The constraints have a form:

$$e^{\mu|}{}_{\alpha(2)} h_{\mu|\alpha(n)}^n = 0, \quad n = 1, \ldots, N - 2. \tag{19}$$

The equations of motion are analogous to the equations of motion for the bosonic fields:

$$\xi_n \varepsilon^{\mu\nu\sigma} \partial_\nu h_{\sigma|\alpha(n)}^n + \varepsilon m a_n \varepsilon^{\mu\nu\sigma} e_{\nu|\alpha}{}^\beta h_{\sigma|\alpha(n-1)\beta}^n + (1 - \delta_{n,1}) m b_n \varepsilon^{\mu\nu\sigma} e_{\nu|\alpha(2)} h_{\sigma|\alpha(n-2)}^{n-2} -$$

$$(1 - \delta_{n,N}) m b_{n+2} \varepsilon^{\mu\nu\sigma} e_{\nu|}{}^{\alpha(2)} h_{\sigma|\alpha(n+2)}^{n+2} - (1 - \delta_{n,N}) m e^{\mu|\alpha(2)} \lambda_{\alpha(n+2)}^{n+2} = 0. \tag{20}$$

Analogously to the bosonic case we contract the equations (20) with ∂_μ and exclude the derivatives of all functions h_σ^n with the aid of (20). In the resulting equations at fixed n the coefficients in front of h_σ^{n+4} and h_σ^{n-4} vanish identically while the term with h_σ^{n-2} vanishes as a consequence of the constraints (19)‡. As a result one finds after some transformations:

$$\varepsilon b_3 \left(\xi_1 a_1 - \frac{5}{3} \xi_3 a_3 \right) e^{\sigma|\alpha(2)} h_{\sigma|\alpha(3)}^3 + \left(2\xi_1 a_1^2 - \frac{4}{3} \xi_3 b_3^2 \right) e^{\sigma|}{}_\alpha{}^\beta h_{\sigma|\beta}^1 - \frac{1}{m} e^{\mu|\alpha(2)} \partial_\mu \lambda_{\alpha(3)}^3 = 0, \tag{21}$$

$$\varepsilon b_{n+2} \left(\xi_n a_n - \frac{n+4}{n+2} \xi_{n+2} a_{n+2} \right) e^{\sigma|\alpha(2)} h_{\sigma|\alpha(n+2)}^{n+2} +$$

$$\left(\frac{2}{n} \xi_n a_n^2 + 2\xi_{n-2} b_n^2 - \frac{2n(n+3)}{(n+1)(n+2)} \xi_{n+2} b_{n+2}^2 \right) e^{\sigma|}{}_\alpha{}^\beta h_{\sigma|\alpha(n-1)\beta}^n -$$

$$2\xi_{n-2} b_n \lambda_{\alpha(n)}^n - \frac{1}{m}(1 - \delta_{n,N}) e^{\mu|\alpha(2)} \partial_\mu \lambda_{\alpha(n+2)}^{n+2} = 0, \quad 3 \leq n \leq N. \tag{22}$$

Similarly to the bosonic case we define $b_{N+2} = 0$. Let us require that the coefficient in front of h_σ^3 in the equation (21) and the coefficients in front of h_σ^{n+2} and h_σ^n in the equations (22) vanish:

$$\xi_n a_n - \frac{n+4}{n+2} \xi_{n+2} a_{n+2} = 0, \quad 1 \leq n \leq N,$$

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‡ In fact, the term with h_σ^{n-2} vanishes without using the constraints because the coefficient in front of h_σ^{n-2} vanishes simultaneously with the coefficient in front of h_σ^{n+2}.

$$\frac{1}{n}\xi_n a_n^2 + \xi_{n-2}b_n^2 - \frac{n(n+3)}{(n+1)(n+2)}\xi_{n+2}b_{n+2}^2 = 0, \quad 3 \le n \le N.$$

These equations are analogous to those in the bosonic case and have the same solutions. Let us choose a normalization $a_N = (-1)^{(N+1)/2}$. In addition in the section 5 it will be shown that the energy boundness conditions along with the requirement of positivity of norms of states demand $\xi_N = (-1)^{(N+1)/2}$. As a result we have:

$$\xi_n = (-1)^{(n+1)/2}, \quad a_n = (-1)^{(n+1)/2}\frac{N+2}{n+2}, \quad b_n = \frac{(-1)^{(n+1)/2}}{2}\sqrt{\frac{(N+2)^2 - n^2}{n(n+1)}}. \qquad (23)$$

Similarly to the bosonic case, from the equations (22) it follows that all Lagrangian multipliers λ^n vanish while the equation (21) takes a form:

$$e^{\mu|}{}_\alpha{}^\beta h^1_{\mu|\beta} = 0. \qquad (24)$$

Let us now turn to the equations for h^n_μ.

Let $N = 1$.

For that case, the Lagrangian multipliers are absent while $h^1_{\mu|\alpha}$ satisfies the equation

$$\varepsilon^{\mu\nu\sigma}\partial_\nu h^1_{\sigma|\alpha} + \varepsilon m\varepsilon^{\mu\nu\sigma}e_{\nu|\alpha}{}^\beta h^1_{\sigma|\beta} = 0.$$

This equation and its consequence (24) are equivalent to the equations (4), (5).

Let $N \ge 3$.

From the constraint (19) at $n = 1$ and the equation (24) it follows that $h^1_{\mu|\alpha} = 0$. Then, similarly to the bosonic case, one proves that $h^n_\mu = 0$, $3 \le n \le N - 2$. The equation of motion (20) at $n = N - 2$ gives the condition (14) for h^N_μ which is equivalent to (4), while the equation of motion (20) at $n = N$ gives

$$\varepsilon^{\mu\nu\sigma}\partial_\nu h^N_{\sigma|\alpha(N)} + \varepsilon m\varepsilon^{\mu\nu\sigma}e_{\nu|\alpha}{}^\beta h^N_{\sigma|\alpha(N-1)\beta} = 0, \qquad (25)$$

from which it follows that h^N_μ satisfies the Klein-Gordon equation with the mass m and describes spin $\varepsilon(N+2)/2$.

Let us note that if the factors of i and ε are absorbed by a redefinition of the coefficients ξ_n, a_n and b_n, then the parameters of bosonic and fermionic actions can be written in a unique form:

$$\tilde{\xi}_n = \zeta_n\xi, \quad \tilde{a}_n = \zeta_n\theta\frac{N+2}{n+2}, \quad \tilde{b}_n = \frac{\zeta_n}{2}\sqrt{\frac{(N+2)^2 - n^2}{n(n+1)}},$$

$$\zeta_n = (-1)^{(n+2)/2} = \exp(i\pi\frac{n+2}{2}),$$

where

$$\tilde{a}_n = i\epsilon a_n, \quad \tilde{\xi}_n = i\xi_n, \quad \tilde{b}_n = ib_n$$

for the fermionic case and

$$\tilde{a}_n = a_n, \quad \tilde{\xi}_n = \xi_n, \quad \tilde{b}_n = b_n$$

for the bosonic case. According to the physical arguments explained in the section 5, $\theta = 1$ in the bosonic case and $\xi = 1$ in the fermionic case.

5 Hamiltonian Formalism

In this section we consider the limitations on the parameters of the model under consideration which follow from the requirements of the positivity of energy and positive definiteness of norms of states. As shown in the previous sections, the fields h_μ^n, $n < N$, all Lagrangian multipliers λ^n and some of the components of the field h_μ^N vanish as a consequence of the field equations. This implies that this theory possesses constraints. Since it is non-degenerate (no gauge symmetry) the constraints can be only of second class. As shown in [21] insertion of second class constraints into an original action gives raise to some equivalent action. Let us use this fact. Also, due to the Lorentz invariance, it is enough for our purpose to consider the components of h_μ^N at vanishing momentum (in the rest frame). The equations of motion (15), (25) along with the constraints (14) (or, equivalently, (16)) give at $\mu = 0$:

$$h_{0|\alpha(N)}^N = 0. \tag{26}$$

Let us introduce a field $(1/2)e_{\alpha_1\alpha_2}^{\mu|}h_{\mu|\alpha_3...\alpha_{N+2}}^N \equiv \tilde{h}_{\alpha(N+2)}^{N+2}$, which is symmetric with respect to all $N + 2$ spinorial indices due to the condition (14) (or, equivalently, (4)). The equation (26) implies that the field $\tilde{h}^{N+2}$ is traceless (we use a realization of the matrices e_μ with $e_{0|\alpha\beta} = \delta_{\alpha\beta}$):

$$\tilde{h}_{11\alpha(N)}^{N+2} + \tilde{h}_{22\alpha(N)}^{N+2} = 0.$$

Thus, $\tilde{h}^{N+2}$ has only two independent components:

$$\tilde{h}_{1...1}^{N+2} \equiv 2^{-(N+2)/2}q_1, \quad \tilde{h}_{21...1}^{N+2} \equiv 2^{-(N+2)/2}q_2.$$

Let us express the actions (6) and (18) in terms of the variables q_1 and q_2 (using the relation $h_{\mu|\alpha(N)}^N = e_{\mu|}^{\alpha(2)}\tilde{h}_{\alpha(N+2)}^{N+2}$).

Bosonic case.

Using $\xi_N = (-1)^{(N+2)/2}\xi$, $a_N = (-1)^{(N+2)/2}\theta$, after simple transformations one gets

$$S_{\mathrm{B}} = \xi q_2 \dot{q}_1 - \frac{\theta m}{2}(q_1^2 + q_2^2).$$

from this expression it follows that ξ can have an arbitrary sign while θ should be positive.

Fermionic case.

Let us set $\xi_N = (-1)^{(N+1)/2}\xi$, $\xi = \pm 1$ and $a_N = (-1)^{(N+1)/2}$:

$$S_{\mathrm{F}} = \frac{i\xi}{2}(q_1\dot{q}_1 + q_2\dot{q}_2) - i\varepsilon m q_1 q_2.$$

It is convenient to pass to a new complex variable η:

$$\eta = \frac{1}{\sqrt{2}}(q_1 + i\xi\varepsilon q_2), \quad q_1 = \frac{1}{\sqrt{2}}(\eta + \eta^\dagger), \quad q_2 = \frac{\xi\varepsilon}{i\sqrt{2}}(\eta - \eta^\dagger).$$

In these terms the action acquires a form:

$$S_{\mathrm{F}} = \xi(i\eta^\dagger\dot{\eta} - m\eta^\dagger\eta).$$

from this expression it follows directly that ε can have an arbitrary sign while for ξ only a value $\xi = 1$ is allowed.

6   Squaring

In the previous sections we have constructed the actions for the irreducible representations of the (2+1)–Poincare group, which can be written in the form

$$S_B = \frac{\xi}{2}\varphi D_N \varphi + \frac{1}{2}\varphi M_N \varphi,$$

in the bosonic case and

$$S_F = \frac{i}{2}\psi D_N \psi + \frac{i\varepsilon}{2}\psi M_N \psi,$$

in the fermionic case, where φ (ψ) denotes a set of all fields h^n_μ, λ^n in the bosonic (fermionic) case, D_N is a homogeneous symmetric (antisymmetric) first order differential operator, M_N is symmetric (antisymmetric) matrix which does not contain derivatives. The matrix M_N is invertible both is the bosonic and in the fermionic cases (while the operator D_N is not). It is natural to expect that, at least in the bosonic case, the operators $D_N \pm M_N$ are square roots of some second order differential operator which describes a field with two values of spin. As mentioned in the Introduction, such operators arise naturally by a reduction of the four-dimensional gauge actions.

Here we formulate a bosonic action which is of the second order in derivatives and describes spin $\pm s$ fields, such that the second order operator is obtained by squaring of the first order operator. Let us start with the action

$$S = \frac{1}{2}\varphi(-D_N + M_N)R(D_N + M_N)\varphi = -\frac{1}{2}\varphi D_N R D_N \varphi + \frac{1}{2}\varphi M_N R M_N \varphi, \qquad (27)$$

where R is some non-degenerate symmetric matrix. The equations of motion have a form

$$(-D_N + M_N)R(D_N + M_N)\varphi = 0,$$

provided that the following condition is true:

$$M_N R D_N - D_N R M_N = 0. \qquad (28)$$

It is easy to see that for this case the action (27) describes the fields of two spins $\pm(N+2)/2$. The condition (28) can be satisfied by choosing a matrix R to be of the form

$$R = M_N^{-1}.$$

Then the action (27) takes a form:

$$S = -\frac{1}{2}\varphi D_N M_N^{-1} D_N \varphi + \frac{1}{2}\varphi M_N \varphi, \qquad (29)$$

and one can say that the first order operators in the action (6) (and (18)) are obtained by "extracting a square root" of the second order operator in the action (29).

An equivalent action can be obtained by introducing the auxiliary fields ω of the same type as φ:

$$S' = \frac{1}{2}\varphi M_N \varphi + \frac{1}{2}\omega M_N \omega + \omega D_N \varphi.$$

Performing in this action a field redefinition:

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2), \quad \omega = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2),$$

one obtains

$$S' = \frac{1}{2}\varphi_1(D_N + M_N)\varphi_1 + \frac{1}{2}\varphi_2(-D_N + M_N)\varphi_2.$$

Thus the field φ_1 describes spin $(N+2)/2$, while the field φ_2 describes spin $-(N+2)/2$.

Let us note that, in the fermionic case, the action

$$S = -\frac{\xi}{2}\psi D_N M_N^{-1} D_N \psi + \frac{\xi}{2}\psi M_N \psi, \quad \xi = \pm 1,$$

which is a counterpart of the bosonic action (29), also describes two spins $\pm(N+2)/2$. However, for any ξ it is not acceptable by physical arguments. The simplest way to see this is to pass to an equivalent action

$$S' = \frac{\xi}{2}\psi M_N \psi + \frac{\xi}{2}\eta M_N \eta + \eta D_N \psi = \frac{\xi}{2}\psi_1(D_N + M_N)\psi_1 + \frac{\xi}{2}\psi_2(-D_N + M_N)\psi_2.$$

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \quad \eta = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2).$$

It is clear that for any value of ξ the kinetic term for one of the fermions has a wrong sign.

Let us mention that such a simple decomposition of the action for two polarizations into a sum of actions for independent components turns out to be possible due to introducing an adequate set of auxiliary fields found in this paper. Note that the fields ω (more precisely their part associated with the 1-forms $h^n_{\mu|\alpha(n)}$) can be interpreted as counterparts of the Lorentz connection in the triadic formulation of gravity.

7   Concluding Remarks

An additional reason why the dynamical systems described in the paper might be of interest is due to their close relationship with the self-duality equations in 2+2 dimensions

$$\varepsilon_{\rho\mu}{}^{\nu\sigma}\partial_\nu h^N_{\sigma|\alpha(N)} = \partial_\rho h^N_{\mu|\alpha(N)} - \partial_\mu h^N_{\rho|\alpha(N)}, \tag{30}$$

which reduce to (5) by virtue of the "compactifying" substitution

$$h^N_{3|\alpha(N)} = 0, \qquad \partial_3 h^N_{\mu|\alpha(N)} = \pm m h^N_{\mu|\alpha(N)}, \tag{31}$$

under condition that 3–components of vectors correspond to a second time-like direction. Similarly, the Euclidian version of (5) can be obtained from the Euclidian version of (30).

Let us note that the set of fields used in the present paper $h^n_{\mu|\alpha(n)}$ $(0 \leq n \leq N)$ corresponds to the decomposition of the fields $h^N_{\mu|\alpha(N/2)\dot\alpha(N/2)}$ (N even) and $h^N_{\mu|\alpha((N+1)/2)\dot\alpha((N-1)/2)}$ (N-odd), which were used in [8] for the description of four-dimensional massless fields of spin $s = N/2 + 1$, into irreducible representations of the diagonal (2+1)–Lorentz group acting both on dotted and undotted spinor indices. Also it is worth mentioning that the Lagrange multipliers $\lambda^n_{\mu|\alpha(n)}$ have a similar structure in spinor indices ($N \geq n > 0$). This suggests an idea that one can try to identify them with a component of the 3+1- dimensional gauge field along the "extra" time within the dimensional reduction of some hypothetical self-dual theory in 2+2 - dimensional space-time.

Self-dual equations of motion of $D = 4$ massless fields of an arbitrary spin can be formulated not only on the linearized level but on the non-linear level too [22]. Recently, supersymmetric self–dual equations for the fields of arbitrary spin have been studied in [23]. In the context of results obtained in this paper it would be interesting to come back to the question of a possibility of formulation of a theory of self-dual fields of an arbitrary spin at the action level.

Also it would be interesting to find a topologically massive version of the 2+1 - dimensional theory of massive fields of an arbitrary spin thus generalizing the previously constructed theories for $s = 1$ [13, 14], $s = 3/2$ [15], $s = 2$ [14] and $s = 5/2$ [19].

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